

A Quantum Theory Of Riemann's Rotors

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Abstract— The angular momentum, angular velocity, Kelvin circulation, and vortex velocity vectors of a quantum Riemann rotor are proven to be either (1) aligned with a principal axis or (2) lie in a principal plane of the inertia ellipsoid. In the second case, the ratios of the components of the Kelvin circulation to the corresponding components of the angular momentum, and the ratios of the components of the angular velocity to those of the vortex velocity are analytic functions of the axes lengths.

Index Terms— Astrophysics; Inglis's Cranking Formula; Kelvin Circulation Vectors; Riemann's Fluids Model; Riemann's Theorem; Quantum Physics; Quantum Tilted Rotors; Vortex Velocity.

1 INTRODUCTION

A classical Riemann rotor is a uniform density fluid with an ellipsoidal boundary and a velocity field that is a linear function of position. Riemann fluids model rotating stars and galaxies [1], [2], spinning gas clouds, [3] and rotating nuclei [4], [5]. Since linear velocity fields span the dynamical continuum from rigid rotation to irrotational flow, the Riemann model is sufficiently general to model most collective rotational systems. The vector observables that measure the character of the rotation are the angular momentum \vec{L} and the Kelvin circulation \vec{C} [5], [6], [7], [8] the vector variables conjugate to \vec{L} and \vec{C} are the angular velocity $\vec{\omega}$ and the vortex velocity $\vec{\lambda}$. Tilted nuclear rotors for which $\vec{\omega}$ is not aligned with a principal axis are a topic of continuing interest [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20].

2 A QUANTUM THEORY OF RIEMANN'S ROTORS

Recently, a quantum theory of Riemann rotors was formulated by simultaneous angular and vortex cranking of the nuclear mean field Hamiltonian:

$$H_{\vec{\omega}\vec{\lambda}} = H_0 - (\vec{\omega}\vec{L}) + (\vec{\lambda}\vec{C}) \quad (1)$$

where the mean field is approximated by the anisotropic oscillator potential:

$$H_0 = -\frac{\hbar^2}{2m}\Delta + \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2) \quad (2)$$

Inglis's cranking formula determines the collective energy of an A-nucleon system [21], [22]:

$$T(\vec{\omega}, \vec{\lambda}) = \sum_{ph} \frac{\left| \langle p | \vec{\omega}\vec{L} - \vec{\lambda}\vec{C} | h \rangle \right|^2}{\epsilon_p - \epsilon_h} \quad (3)$$

Self-consistency of the mean field with the density distribution requires that:

$$\omega_x N_x = \omega_y N_y = \omega_z N_z \quad (4)$$

where $N_k = \sum (n_k + 1/2)$ denotes the total number of quanta in the k^{th} direction. The following semiclassical correspondence theorem has been established: [23] At self-consistency, the "Inglis" collective energy, Eq. (3), equals the classical value for the kinetic energy of a Riemann rotor [1]:

$$T(\vec{\omega}, \vec{\lambda}) = \frac{I_0}{4} \sum_{k=1}^3 \left[(a_i^2 + a_j^2)(\omega_k^2 + \lambda_k^2) - 4a_i a_j \omega_k \lambda_k \right] \quad (5)$$

where $i; j; k$ are cyclic, $a_1; a_2; a_3$ are the axes lengths of the inertia ellipsoid in units of R , and the moment of inertia of a sphere of radius R and total mass $M = mA$ is $I_0 = (2/5)MR^2$. If the nuclear volume is set equal to the spherical value $4\pi R^3/3$, then the product of the dimensionless axes lengths equals unity, $a_1 a_2 a_3 = 1$. Note that if the vortex velocity vanishes, then the self-consistent collective energy equals the rigid body energy, a well-known result [24]. In addition, the quantum expectations of the angular momentum and Kelvin circulation equal their classical Riemann rotor values [1]:

$$\begin{aligned} \langle L_k \rangle &= \frac{I_0}{2} \left[(a_i^2 + a_j^2) \omega_k - 2a_i a_j \lambda_k \right] \\ \langle C_k \rangle &= \frac{I_0}{2} \left[2a_i a_j \omega_k - (a_i^2 + a_j^2) \lambda_k \right] \end{aligned} \quad (6)$$

These expectations are given by derivatives of the kinetic energy with respect to the angular velocity and the vortex velocity [25]:

$$\begin{aligned} \langle L_k \rangle &= \left(\frac{\partial T}{\partial \omega_k} \right) \langle C_k \rangle \\ &= - \left(\frac{\partial T}{\partial \lambda_k} \right) \end{aligned} \quad (7)$$

The collective energy may be expressed as:

$$T(\vec{\omega}, \vec{\lambda}) = \frac{1}{2}(\vec{\omega} \langle \vec{L} \rangle - \vec{\lambda} \langle \vec{C} \rangle) \quad (8)$$

The energy in the rotating intrinsic frame for ordinary “Inglis” cranking of the angular velocity vector is minimized with respect to orientation when the angular velocity and angular momentum vectors are parallel [11], [12], [20]. Because the vortex velocity is independent of the angular velocity, minimization of the intrinsic energy $\tilde{E}(\vec{\omega}, \vec{\lambda}) = \langle H_{\vec{\omega}\vec{\lambda}} \rangle$ with respect to the orientation of $\vec{\lambda}$ also requires that the vortex velocity and the Kelvin circulation vectors are parallel:

$$\begin{aligned} \vec{\omega} \times \langle \vec{L} \rangle &= 0 \\ \vec{\lambda} \times \langle \vec{C} \rangle &= 0 \end{aligned} \quad (9)$$

This is proven by computing the change in the intrinsic energy when the orientation of the vortex velocity is shifted infinitesimally from $\vec{\lambda}$ to $\vec{\lambda} + \delta\vec{\lambda} = \vec{\lambda} + \varepsilon \hat{n} \times \vec{\lambda}$ where \hat{n} is an arbitrary unit vector:

$$\begin{aligned} \tilde{E}(\vec{\omega}, \vec{\lambda} + \delta\vec{\lambda}) - \tilde{E}(\vec{\omega}, \vec{\lambda}) &= \varepsilon (\hat{n} \times \vec{\lambda}) \cdot \langle \vec{C} \rangle + O(\varepsilon^2) \\ &= \varepsilon \hat{n} \cdot (\vec{\lambda} \times \langle \vec{C} \rangle) + O(\varepsilon^2) \end{aligned} \quad (10)$$

Hence, at equilibrium, $\hat{n} \cdot (\vec{\lambda} \times \langle \vec{C} \rangle) = 0$ for all directions \hat{n} , or $\vec{\lambda} \times \langle \vec{C} \rangle = 0$.

3 CONCLUSION

Riemann [26] proved in 1860 that the classical rotors in equilibrium fall into three classes:

1. Rigid rotors ($\vec{\lambda} = 0$) which encompasses the Maclaurin spheroids and the Jacobi triaxial ellipsoids;
2. S-type ellipsoids for which the directions of $\vec{\omega}$ and $\vec{\lambda}$ are aligned with a principal axis; and
3. Tilted ellipsoids for which the directions of $\vec{\omega}$ and $\vec{\lambda}$ lie in a principal plane.

The case where $\vec{\omega}$ and/or $\vec{\lambda}$ do not lie in a principal plane is specifically excluded. It is remarkable that Riemann's theorem is also true for quantum cranked Riemann rotors. To prove the theorem, substitute the explicit formulae for the expectations of the angular momentum and Kelvin circulation, Eq. (6), into the parallelism conditions, Eqs. (9). If the angular

velocity vector $\vec{\omega}$ is neither aligned with a principal axis nor lies in a principal plane, then $\omega_1, \omega_2, \omega_3$ are all nonzero and the parallelism conditions constitute a set of six simultaneous equations in the three unknown ratios, $\lambda_1/\omega_1, \lambda_2/\omega_2$ and λ_3/ω_3 . It can be shown that only four of these equations are independent. Therefore, this simultaneous system is overdetermined for the three unknown ratios. Since the assumption that all three components ω_k are nonzero implies a contradiction, one of the angular velocity components must vanish, say ω_1 . But, writing out the y-component of Eq. (9), $0 = \omega_3 \langle L_1 \rangle = -2a_2 a_3 \omega_3 \lambda_1$, one concludes that λ_1 must also vanish. Hence, $\vec{\omega}$ and $\vec{\lambda}$ lie in a principal plane of the inertia ellipsoid. If ω_2 and ω_3 are both nonzero, then the parallelism conditions produce two independent equations in two unknown ratios whose solution is:

$$\begin{aligned} \frac{\lambda_2}{\omega_2} &= \frac{(4a_1^2 - a_2^2 + a_3^2 \pm q)}{4a_1 a_3} \\ \frac{\lambda_3}{\omega_3} &= \frac{(4a_1^2 - a_2^2 + a_3^2 \pm q)}{4a_1 a_2} \end{aligned} \quad (11)$$

where:

$$q^2 = (4a_1^2 - (a_3 + a_2)^2)(4a_1^2 - (a_3 - a_2)^2) \quad (12)$$

If only one of the components of $\vec{\omega}$ is nonzero, then the angular velocity, vortex velocity, angular momentum, and Kelvin circulation vectors are all aligned with a single principal axis. The ratio λ/ω is undetermined for such an S-type ellipsoid. This completes the proof of Riemann's theorem.

Since q cannot be imaginary, there are only three types of tilted rotors. Choosing an ordering of the axes lengths in the principal plane, say $a_3 \geq a_2$, yields the following:

- **Type I:** $2a_1 \geq a_3 + a_2$;
- **Type II:** $2a_1 \leq a_3 - a_2$ and $a_2 \leq a_1$;
- **Type III:** $2a_1 \leq a_3 - a_2$ and $a_1 \leq a_2$

Type II and Type III tilted rotors are ultradeformed prolate-like solutions for which the ratio of the longest to the shortest axis is at least three to one. Type I solutions are tilted oblate-like rotors; these have been studied in the classical macroscopic approximation where the potential energy is a sum of the attractive surface energy plus the repulsive Coulomb potential [5].

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